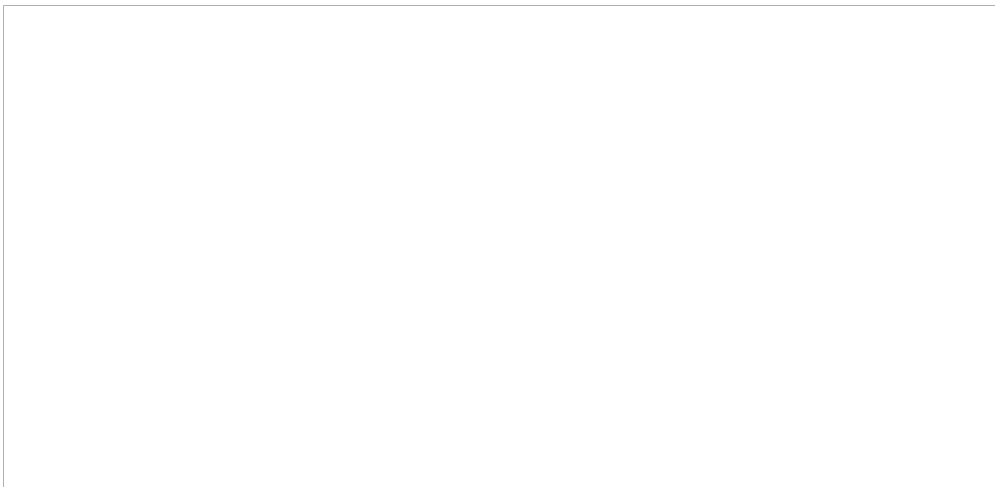


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THEORY OF SHOCK WAVES AND INTRODUCTION
TO GASDYNAMICS

Ya. I. Zel'dovich
 [Refer to 00-W-12268 for complete table of Contents]

VII Theory of the shock wave. Introduction

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In the section on the propagation of sound we have shown that ~~an~~ acoustic wave must deform during its propagation. The "crests of the waves", i.e. those points at which the material is compressed and moves in the direction of propagation of the wave, go forward. On the other hand the "troughs", i.e. regions of rarefaction where the velocity of motion has a direction opposite to that of the propagation of sound, all lag behind the wave. Thus, in deforming, the acoustic wave appears to be curving over. This phenomenon is analogous to the one observed ^{when} ~~where~~ the ocean waves run against a sloping shore.

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taneous increase in the gradients and for a spontaneous formation of discontinuities during compression.

In the theory of flow through a Laval nozzle we have shown that it is impossible, by merely using the equations for a continuous flow with constant entropy, to describe a series of intermediate processes in a large region of back pressures.

Finally this limitation of classical gasdynamics was particularly well outlined in the last problem we considered, concerning the motion of a gas produced by a sudden movement of a piston. In this case if the piston moves in the direction of the gas, $w > 0$, the differential equations of gasdynamics lead to absurd triple-valued solutions, i.e. to solutions which give three pressures, three temperatures and three velocities at a given point simultaneously.

All the cases we have mentioned clearly show the necessity for finding other forms of solutions which do not follow directly from the equations of gasdynamics of ideal gases (we use "ideal" in the sense that viscosity and heat conductivity are absent).

We may expect the unknown processes to be characterized by large gradients so that, to a certain approximation, we may treat them as the propagation of velocity, pressure and density discontinuities - the so-called shock waves.

Before presenting the history of the problem of shock waves we shall present an elementary derivation of the equation of a shock wave in very much the same manner as done by Hugoniot in his well-known memoirs "On the propagation of a discontinuity" [56], assuming beforehand that the discontinuity exists and not considering the question of its actuality, stability etc.

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XI Graphical method of treating the theory of shock waves.

Waves near the critical point.

A very convenient aid for a simple and visual analysis of the theory of shock waves is a representation of the processes and states on a diagram whose axis of the abscissas represents the specific volume v and whose axis of the ordinates represents the pressure p . It has already been pointed out that for every given initial point (point A, p , v , in Figure 27) there is one definite Hugoniot curve. We shall show in Figure 27 how to determine graphically the propagation velocity of a shock wave. We shall make use of the equation which gave us the velocity of the shock wave as a function of the pressures and specific volumes before and after compression:

$$D^2 = u_1^2 = v_1^2 \frac{p_2 - p_1}{v_1 - v_2} \quad (\text{XI} - 1)$$

For a given initial state p_1 , v_1 , of the substance, the term v_1^2 before the fraction is a constant and the propagation velocities of the shock waves, corresponding to various degrees of compression, depend on the ratio $p_2 - p_1 / v_2 - v_1$, i.e. on the slope of the corresponding lines which cannot point p_1 , v_1 with points representing the state p_2 , v_2 after compression. Thus it is quite clear from the drawing that point C (which corresponds to a greater pressure than point B) corresponds to a shock wave which propagates with a higher velocity since the slope of the line AC is greater than the slope of the line AB. It is quite important that equation (XI - 1) was derived

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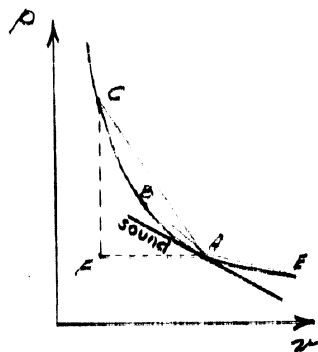


Fig. 27 Propagation velocity of the shock wave is determined by the slope of the chord, for example, AC, AB, AE; the velocity of sound is given by the slope of the tangent.

by us from only two initial equations (equation of continuity of mass and equation of continuity of momentum) independently of the equation of continuity of energy. Therefore it will be valid for all cases in which the equation of continuity of momentum is not violated, i.e. when there are no external forces such as the friction of gas against the walls.

For all these cases (in particular for cases when we have present chemical reaction or external sources of heat and energy which affect only the energy equation and not the equation of momentum) expression (XI-1) remains valid. In particular, expression (XI-1) also pertains to the propagation of detonation in explosive gaseous mixtures [8, 59, 60].

We should particularly note that equation (XI-1) is obtained by constructing the equations of conservation of mass and momentum for only the initial and final state of the gas in the wave. By making use of lines AC or AB for computing the velocity we do not imply that the intermediate states (see Figure 236) are represented by the points on these lines.

If we are interested in the intermediate state of compression inside a ^h shock, detonation or other wave front which propagates steadily through a gas, then in addition to external forces we must also take

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into account the internal viscous forces of the gas which drop out when we compare the initial and final state. If for any reason we could neglect the effect of viscosity and internal friction ~~the~~ equation (XI-1) could be applied to all intermediate states through which the substance ^Spasses from its initial to its final state. Such is the situation in a detonation wave where the width of the wave depends on the velocity of chemical reaction and, in general, is considerable so that the effect of viscosity is negligible. This question is considered in detail and a complete bibliography is presented in the works of the author [8] [103].

It is also easy to find the graphical representation of the velocity of sound in Figure 27. We obtain the propagation of sound as a limiting case of the propagation of very weak shock waves. Thus the velocity of propagation of sound on the diagram in Figure 27 will be given by the limiting position of the slope of the secant where the second point, corresponding to the final state of the substance, approaches the first point. In other words, the velocity of sound will be given by the slope of the line which is tangent to the Hugoniot curve at a point which represents the initial state of the substance considered.

Comparing expression (XI-1) for a small $p_2 - p_1$ with the expression for the velocity of sound $c^2 = -v^2 \frac{\partial p}{\partial v}_S$, we conclude that at the initial point A, the Hugoniot curve touches a line of constant entropy. (Poisson's Curve)

It follows from the drawing that for an ideal gas with constant specific heat whose Hugoniot curve is shown in Figure 27, the velocity

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of propagation of the shock wave is greater than the velocity of propagation of sound in the initial gas $D = u_1 > c_1$. In the limit, by increasing the pressure of the shock wave indefinitely, we can get as high velocity as we choose for the propagation of the shock wave. On the other hand for a rarefaction wave, whose final state E (Figure 27) lies below the initial state, we would get a propagation velocity less than the velocity of sound. If we construct Poisson's curve through the final state of the compressed gas, for example through the point B, we can obtain a relation between the velocity of the shock wave and the velocity of sound in the compressed gas. The velocity of propagation of the wave with respect to the compressed gas is:

$$u_2^2 = (D - u)^2 = v_2^2 \frac{\rho_1 - \rho_2}{v_2 - v_1}, \quad (XI-2)$$

This expression is quite similar to the expression for the velocity of the wave with respect to the initial gas. In Figure 28 the Hugoniot curve H_B is constructed through point B which is taken as the initial state. From the symmetry of the equations we see that if B lies on H_A then H_B passes through the point A (see equations 2X-10, 11).¹ At the point B the H_B curve touches Poisson's curve. From the position of lines H_B and BA, in Figure 28, it follows that $c_2 > u_2 = D - u$ so that the velocity of sound in the gas compressed by the wave is greater than the velocity of the wave with respect to the compressed gas.

By means of a $p-v$ diagram we can consider the question of the

¹ H_A and H_B are abbreviations for Hugoniot curves whose initial points are A and B respectively.

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increase of entropy in a shock wave. We shall compare the expression for the variation of the internal energy of the gas in a shock wave with the general thermodynamic expression for the differential of the energy. In a shock wave

$$\Delta E = E_2 - E_1 = \frac{p_1 + p_2}{2} (v_1 - v_2). \quad (XI-3)^2$$

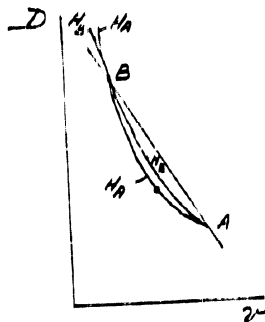


Fig 28. The relation between the velocity of propagation of the wave with respect to the initial state A and the velocity of sound in state A is given by the ratio of the slope of chord AB to the slope of the tangent to curve H_A at point A. The relation between the velocity of propagation of the wave with respect to the compressed substance in state B and the velocity of sound in state B is given by the ratio of the slope of chord AB and the slope of the tangent to curve H_B at point B. A direct comparison of velocities in different states is impossible since the coefficient used to get velocity from slope depends on the specific volume v .

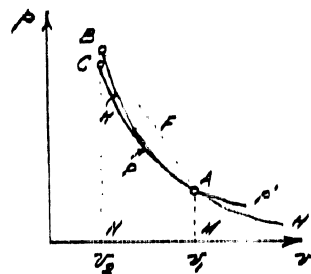


Figure 29. The increase in entropy during compression in shock wave AB depends on the sign and magnitude of the area $AF BCPA$; AHB is the Hugoniot curve; APC is the Poisson curve.

² Equation (XI-3) is obtained from (VIII-6) if we use specific volume in place of density.

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At the same time the general expression is $dE = T dS + p dv$. If the volume varies within the same limits, then, along the Poisson curve (isentropes), we have

$$\begin{cases} dS = 0 \\ E_2 - E_1 = \Delta E = \int_{v_1}^{v_2} p dv \end{cases} \quad (XI-4)$$

If we compare the expression for the variation of entropy along the Poisson curve (P) with the expression for the variation of entropy, during shock compression, along the Hugoniot curve (H) we shall obtain an expression for the variation ΔS in entropy during shock compression:

$$\bar{T} \Delta S = \frac{p_1 + p_2}{2} (v_1 - v_2) - \int_{v_2}^{v_1} p dv \quad (XI-5)$$

Integrals (XI-4) and (XI-5) are taken along the Poisson curve.

Let us examine, by means of Figure 29, the relation between two terms of the last equation.

In this curve APC is the Poisson curve (isentropes) and AHB is the Hugoniot curve. The variation in entropy during compression by the shock wave is equal to $S_B - S_A = S_B - S_C$ and, in accordance with equation (XI-5), depends on the difference between area AFBHM and area APCNM. The product of the absolute temperature¹ and the entropy increment is equal to the difference between these areas, i.e. to the area of figure APCBF.

1

The value of \bar{T} in equation (XI-5) lies between T_C and T_B . To prove this we go from state A into state B (Figure 29) by isentropic compression (AC) and a subsequent heating of the gas at constant volume (CB).

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We divide this area into two parts by the line AC. The first part is a section whose end points A and C are bounded by the segment APC of the Poisson curve and by the chord AC. The second part is the triangle ABC.

We write the equation in the following form, denoting by F the area of the figures:

$$\overline{AS} = F_{\text{seg. APC}} + F_{\text{triangle ABC}}. \quad (\text{XI-6})$$

It is easy to find the area of the triangle. We take the segment BC as the base. Then the height will be $v_1 - v_2$. The length of the segment BC in the p, v plane is $\left(\frac{\partial p}{\partial S}\right)_v \Delta S$, and the length of the triangle is

$$\frac{1}{2} \left(\frac{\partial p}{\partial S}\right)_v (v_1 - v_2) \Delta S.$$

Substituting in the initial equation we get

$$\overline{AS} = F_{\text{seg}} + \frac{1}{2} \left(\frac{\partial p}{\partial S}\right)_v (v_1 - v_2) \Delta S, \quad (\text{XI-7})$$

$$\Delta S = \frac{F_{\text{seg}}}{\overline{S} - a}, \quad (\text{XI-8})$$

where $a = \frac{1}{2} \left(\frac{\partial p}{\partial S}\right)_v (v_1 - v_2)$ and the correction due to the area of the triangle is small. If $\Delta S \sim (v_1 - v_2)^n$ then the area of the triangle $\sim \Delta S (v_1 - v_2) \sim (v_1 - v_2)^{n+1}$ is an infinitesimal of a higher order with respect to ΔS and consequently of higher order with respect to the area of the segment.

Therefore the sign of the entropy change is determined completely by the sign of the area of the segment, i.e. by the mutual position of the Poisson curve and its secant. This in turn depends on the convexity or concavity of the Poisson curve, i.e. on the sign of the derivative $\left(\frac{\partial^2 p}{\partial v^2}\right)_S$. If a approaches \overline{S} , then $\Delta S \rightarrow \infty$.

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This actually takes place in an ideal gas for $v_2 \rightarrow \frac{k-1}{k+1} v_1$, when $p \rightarrow \infty$ on the Hugoniot curve. $\bar{T} < a$ corresponds to a negative pressure and to similar conditions which have no physical sense in the present case.

For weak waves it is now easy to find the laws governing the changes of entropy in a shock wave. We shall carry out the calculations by expanding all expressions in series of $\Delta v = v - v_1$, and retaining everywhere only the major term which gives a finite result different from zero.

The equation for the Poisson curve is

$$p = p_1 + \left(\frac{\partial p}{\partial v}\right)_{s,1} \Delta v + \left(\frac{1}{2} \frac{\partial^2 p}{\partial v^2}\right)_{s,1} (\Delta v)^2 \quad (\text{XI-9})$$

The second subscript indicates that the values of the derivatives are taken for state 1 (point A Figure 29).

Letting $\Delta v_2 = v_2 - v_1 = \omega$, we find ¹ the pressure p_2' at the point C (Figure 29). Dropping the subscripts we have:

$$p_2' = p_1 + \frac{\partial p}{\partial v} \omega + \frac{1}{2} \frac{\partial^2 p}{\partial v^2} \omega^2 \quad (\text{XI-10})$$

We write the expression for the change in entropy, neglecting the area of the triangle ABC in (XI-5, 6, 7):

$$\begin{aligned} T \Delta S &= \frac{p_1 + p_2'}{2} (-\omega) - \int_{v_1}^{v_2} p dv = \frac{1}{2} \left(\frac{\partial^2 p}{\partial v^2}\right)_{s,1} \omega^3 \\ &= \frac{1}{2} \frac{\partial^2 p}{\partial v^2} (v_1 - v_2)^3 \end{aligned} \quad (\text{XI-11})$$

Comparing the Hugoniot equation

$$I_2 - I_1 = \left(\frac{v_2 - v_1}{2}\right) (p_2 - p_1) \quad (\text{XI-12})$$

with expression $dI = T dS + v dp$ we can transpose p and v .

¹

We note that $v_2 < v_1$, so that $\omega < 0$

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As a result we get ¹

$$\Delta S = \frac{1}{2} \frac{\partial^2 v}{\partial p^2} (p_2 - p_1)^3 = \frac{1}{2} \frac{\partial^2 p}{\partial v^2} \left(-\frac{\partial v}{\partial p} \right)^3 (p_2 - p_1)^3 \quad (\text{XI-13})$$

In a weak shock wave the change in the entropy is proportional to the cube of the amplitude. At the initial point the Hugoniot curve touches the Poisson curve. At this point the curves have a common tangent and a common radius of curvature (tangency of the second order). The tangency is accompanied by intersection (see the extension of the curves for $v > v_1$ on Figure 29).

These results were first obtained by Joxuet [58] in a more complicated manner without the use of a geometric treatment. Since the more complete work of Joxuet was published before the report of Zemlen [99] (in the second note of Vol. 142 Zemlen "notes" what he should really "quote" from Joxuet) the usual practice of attributing to Zemlen the proof for the impossibility of a rarefaction wave is completely unfair.

By examining (XI-11) we see that for an ideal gas the Poisson curve is everywhere convex ² with respect to the axis of the abscissas.

² An extremely rapid increase in the specific heat is necessary to realize a condition where with a decrease in $\kappa = c_p/c_v$ the absolute value $\left(\frac{\partial p}{\partial v} \right)_S = -\kappa \frac{p}{v}$ would decrease with an increase in temperature.

¹ Perhaps we should point out once again that the calculation of the area of the trapezoid bounded by the straight line A'B (Figure 29) is based on the expression for the Hugoniot curve which follows from the laws of conservation applied to the state before and after the waves. This result is no way connected with the question of how the state actually varies within the wave (see XII).

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This shows that the entropy increases in a compression shock wave. On the other hand in a sharp rarefaction wave, in which the equations of continuity apply, the entropy would increase. Thus we see immediately that the propagation of a rarefaction wave with a thin front (similar to a compression shock wave) in an ideal gas is impossible.

For weak waves when the equation of state is arbitrary we can use Figure 29 to arrive at conclusions concerning the relation between the propagation velocity of a shock wave and the velocity of sound in the substance before and after compression. For the compression to propagate through the gas in the form of a shock wave with a rather steep front, it is necessary that the Poisson curve be convex downwards, i.e. that it have the form shown in Figure 29. However in this case it is clear from geometric considerations, that the slope of the tangent to the curve at point A must be less than the slope of the secant AB. On the other hand the slope of the tangent at point B, which represents the final state, or the slope of the tangent at the point C close to B, is greater than the slope of the secant.¹ Thus we obtain an elementary derivation of the relation first discovered by ^uJouguet according to which the compression propagates in the form of a shock wave if the velocity of sound before compression is less than the velocity of propagation of the shock wave while the velocity of sound in the substance after compression is greater than the velocity of the shock wave with respect to the compressed substance.²

¹ The variation of the quantity $\left(\frac{\partial s}{\partial v}\right)_s$, on which the velocity of sound depends is of the first order in $v_1 - v_2$ during the transition from A to C or from A to B; the variation of $\left(\frac{\partial s}{\partial v}\right)_s$ during the transition from C to B is of the third order.

² For small amplitude the quantities D, C_1 and C_2 differ by a quantity which is proportional to the amplitude. The velocity u is also pro-

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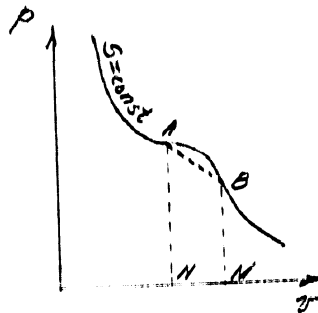


Fig. 30 Poisson curve with a segment having an anomalous convexity upward. Along such a segment rarefaction shock waves would be possible.

In case of Poisson curve with an inverse concavity (Fig. 30 section AB) compression in the shock would be accompanied by decrease in entropy since the area bounded by the Poisson curve is greater than the area bounded by the secant, the vertical lines and the axis of the abscissas. In a substance in which the Poisson curve has a concavity of inverse sign, the compression waves will not be very sharp. A compression produced in some part of the substance, say by the movement of a piston, will gradually expand like the rarefaction wave in an ideal gas considered above. On the other

hand in such a substance the rarefaction wave will propagate with an extremely sharp front whose curvature will not decrease with time and

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portional to the amplitude. For accuracy to amounts proportional to the square of the amplitude, the velocity of the shock wave is equal to the arithmetic mean of the velocity of sound in the initial state and the velocity of propagation of a disturbance in the direction of the wave in the compressed ^{moving} gas $c_2 + u$:

$$D = \frac{c_1 + c_2 + u}{2}$$

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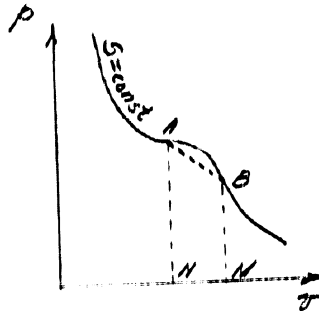


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$$D = \frac{c_1 + c_2 + u}{2}$$

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will be ^{characterized} ~~determined~~ by small values of heat conductivity and viscosity. This corresponds to an inverse relation between the velocity of the shock wave and the velocity of sound. Indeed, in the propagation of a rarefaction wave whose initial state is given by point A and final state by point B (Figure 30), the velocity of propagation of the rarefaction wave AB in state A is determined by the slope of AB and is greater than the velocity of sound in state A. We can see this from the character of the intersection between the curve and the secant at point A, where the tangent to the Poisson curve is more flat than the line AB. On the other hand, at point B, which describes the state of the substance after the sharp rarefaction wave has passed, the velocity of sound is greater than the velocity of a finite disturbance.

Do substances exist in nature for which in some part of the p , v plane the Poisson curve has a convexity directed upwards? We can expect the appearance of such states near the critical point where we have a liquid and a gas. Actually, long before the critical point, the isotherms have an inflexion (at the critical point itself the inflexion of the isotherm also becomes horizontal). For a substance with a sufficiently large molecular specific heat, whose isotherms and adiabatic curves differ by a sufficiently small amount, we can expect that outside the domain of two-phase systems when the substance is completely stable and in one phase, the adiabatic curve will also have an inverse sign for the second derivative. The relation between the structure of the compression wave and the rarefaction wave will become inverse compared with the relation between a sharp compression shock wave and a diffused rarefaction wave in usual gases away from the critical point.

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Figure 31, which represents the p, v plane for the case $C_v = 40$ cal/deg. mole, shows three curves as follows: line II separating the region with $(\frac{\partial^2 p}{\partial v^2})_s < 0$, the adiabatic curve passing through this region and line I separating the crosshatched region of two-phase systems (the last does not depend on C_v). The calculations were carried out with the participation of engineer F. Ye. Yudin (Combustion Laboratory, IKhF).

In the equation of van der Waals the specific heat at constant volume depends only on temperature in the entire domain of single-phase systems. The energy of a homogeneous substance, which satisfies the van der Waals equation, can be represented in the form:

$$E = E_1(T) + E(v) = \int C_v dT - \frac{a}{v}.$$

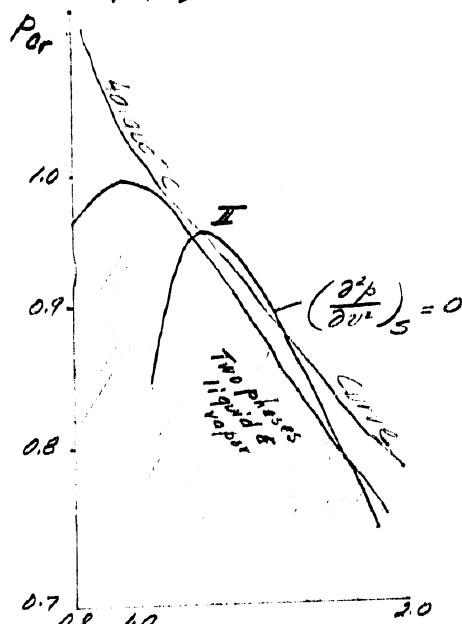


Figure 31 Adiabatic curves with anomalous convexities

in a van der Waals gas with specific heat $C_v = 40$.

The crosshatched area represents two-phase systems.

Curve II separates the region of states with anomalous

convexities of the adiabatic curve. Under curve II

$$(\frac{\partial^2 p}{\partial v^2})_s < 0.$$

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This circumstance considerably simplifies the calculations because the entropy of a van der Waals gas can also be represented in the form of a sum of functions of temperature and specific volume. It would be very interesting to study experimentally the shock waves and rarefaction waves in a gas with a high specific heat in that region where ^{we} can expect the above-mentioned abnormalities. For this purpose we can take an organic compound of high molecular weight which will not decompose at the critical temperature.

The establishment, in a general way, of the relation between the velocity of sound in a substance before and after the passage of the shock wave and the change in entropy in a shock wave is essential and gratifying since it is clear (see the remark of Thomson referenced in the article of Rankine [78]) that the relation between the velocity of the shock wave and the velocity of sound determines the mechanical stability of the wave. It is necessary that the shock wave propagate with a velocity greater than the velocity of sound in the gas, which is subjected to its action, for the disturbance produced by the shock wave not to move forward with sonic velocity. It is necessary that the shock wave propagate with respect to the compressed gas with a velocity less than the sonic velocity in the compressed gas. Only then can we imagine a causative relation between the motion of a piston, which produces a shock wave, and the propagation of a shock wave, since the transfer of the disturbance from the piston to the shock wave front takes place through a layer of compressed gas. We shall meet the same criteria when we study the origin of the shock wave. It is a very satisfying fact that these criteria of the mechanical stability of a shock wave, which can be perceived directly, can be strictly tied to the sign

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of the entropy change in the wave. This sign in general tells us of the possibility or impossibility of propagation of a shock wave which satisfies the equations of continuity of mass, momentum and energy.

The relation between the sign of ΔS and the inequalities, which pertain to the velocity of sound, will be broken only if in the interval of pressure change considered, both signs of $\partial^2 p / \partial v^2$ are present so that the Poisson curve has more than two points of intersection with the straight line. The study of complex processes which have discontinuities and attached diffused waves is beyond the scope of this book.

XII Structure of the shock wave front.

We begin the investigation of that thin layer of the shock wave inside which a transition takes place from one state to another - a layer between the guide lines A and B of Figure 235. In the preceding presentation we did not consider the processes inside this layer on the basis that the thickness of the layer, which is determined by dissipative forces, is very small and the results of the processes in the layer can be obtained from the equations of continuity without a detailed analysis of the processes themselves.

Now we shall be interested precisely in the processes inside the layer and the thickness of the layer. We shall consider separately two limiting cases: 1) the case of low viscosity and 2) the case of low heat conductivity. The case of the combined effect of viscosity and conductivity which is more difficult mathematically (but not physically) will not be considered. For it we shall present only the final expression for the thickness of the transition layer.

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The first case is marked by the fact that equation (XI-1)

$$D^2 = v_1^2 \frac{\rho - \rho_1}{v_1 - v},$$

is valid not only for the final state, which is attained in compression, but for all intermediate states inside the layer.

Indeed this equation follows from the first two continuity equations - continuity of mass and continuity of momentum.

The equation of state of a substance in simple form (VIII-1)

$$\rho u = \frac{u}{v} = \text{const}$$

is always satisfied in the propagation of plane waves. In the propagation of waves in a tube it is necessary that ^{the} cross section of the tube be constant. In addition to this no substance must be consumed or liberated on the walls of the tube. To satisfy the momentum equation (VIII-2)

$$p + \rho u^2 = \text{const}$$

it is necessary that no external forces act on the substance. In considering the propagation in the tube it is necessary to neglect the friction against the walls of the tube. Finally in considering intermediate states, which are of interest to us here, it is necessary (to satisfy (VIII-2) that the forces of internal friction (viscosity) be small.

For a shock wave in a medium where the processes are taken into account by the energy equation, for example the liberation of chemical energy (detonation wave - see [8], [59], [60]) or heat conductivity, equation (XI-1) is applicable to all intermediate states. During the propagation of a shock wave as a whole the velocity, with which each of the intermediate states moves with respect to the original state, is the same. In equation (XI-1) we must consider the quantity D constant. Thus, this equation leads to a linear

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relation between pressure and volume

$$p = p_1 + \frac{D^2}{2} - \frac{D}{2} v. \quad (\text{XII-1})$$

In the p, v plane (Figure 32) the state varies along a straight line connecting the points which describe the initial state A and the final state B of the substance.

If we know the relation between pressure and density which is valid for the entire shock wave, we can find its thickness by elementary integration.

It can be shown that along the straight line AB the entropy reaches a maximum somewhere (point M, Figure 32) midway between the initial and final state of the substance.

Indeed, at point A the velocity of the wave with respect to the substance is greater than the speed of sound; at point B the velocity of the wave is less than the speed of sound. At some point M the velocity of the wave is sonic. At this point the straight line AB is tangent to the Poisson curve and consequently the entropy has a maximum.¹

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Under this assumption of no viscosity the change in entropy occurs only as a result of heat conductivity. In a stationary process, when the coordinates are chosen so that the wave is at rest, we change easily from a particle (material) derivative to a derivative with respect to a coordinate.

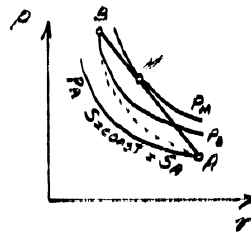


Fig. 32 A is the initial and B the final state of the gas compressed by the shock wave. The unbroken curves are the Poisson curves, i.e. lines of constant entropy increasing from S_A to S_B . In the absence of viscosity but with heat conductivity present, the state changes along the line AB on which the entropy reaches a maximum at point M. In the absence of heat conductivity but with viscosity present, the state changes along the broken line HB on which the entropy increases monotonically from A to B. The Hugoniot curve is not shown on the figure (it also passes through A & B but does not coincide with the broken line).

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In Figure 32 Poisson curves passing through points A, B, M are labelled P_A , P_B , P_M .

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In this case the sign of the partial derivative is also unnecessary since the process considered is stationary in the selected system of coordinates and does not depend on time. Finally:

$$\rho T u \frac{dS}{dx} = \frac{d}{dx} \lambda \frac{dT}{dx} \simeq \lambda \frac{dT}{dx^2}, \quad (\text{XII-2})$$

where λ is the conductivity of the substance. The temperature, at least in a weak shock wave, changes monotonically along the line AB.

The unknown solution - the distribution of temperature and entropy as a function of coordinates - has the form shown on Figure 33. The point at which the entropy reaches a maximum coincides with the inflexion point of the temperature curve (a function of the coordinate)

From the evaluations of the preceeding section it is easy to find the orders of the quantities (assuming the change in volume during compression to be an infinitesimal of the first order): $\Delta p, \Delta T$, are of the first order proportional to Δv ; $S_M - S_A \sim S_H - S_B$ are of the second order proportional to $(\Delta v)^2$. It is easy to evaluate the thickness of the shock wave front by integrating (XII-2) to point M:

$$\frac{1}{v} T u (S_M - S_A) = \lambda \left(\frac{dT}{dx} \right)_M \simeq \lambda \frac{dT}{dx} \quad (\text{XII-3})$$

From our evaluations, it follows that:

$$\Delta x \sim \lambda \frac{dT}{S_M - S_A} \sim \lambda \frac{\Delta v}{(\Delta v)^2} \sim \frac{\lambda}{\Delta v}. \quad (\text{XII-3a})$$

The value of Δx given by the last equation is shown in Figure 33.

The order of magnitude of the coefficient can be evaluated from the dimension

$$\Delta x \sim \frac{\lambda}{R} \frac{v^2}{c \Delta v}, \quad (\text{XII-4})$$

where R is the gas constant. The dimensions of the thermal capacity (specific heat) are cal/deg.gm. The powers of v and c are selected to obtain a quantity whose dimension is length.

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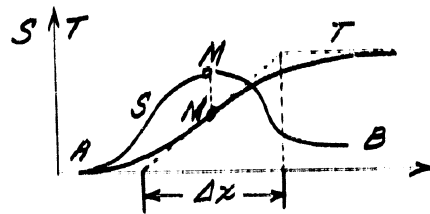


Figure 33. The internal structure of the shock wave of small amplitude with conductivity present but with no viscosity. For definitions see Figure 32.

The distribution shown in Figure 33 represents a concrete form of Rankine's idea [78].

It is curious that in the case of a strong compression a unique fundamental difficulty arises. A temperature maximum is attained on the line AB between the points A and B if the pressure P_B in the shock wave is greater than $1.5 P_A$ (with $c_p/c_v = 7/5$, for a diatomic gas). In addition to this the temperature maximum is at a higher pressure than the entropy maximum.

When a temperature maximum is present it becomes impossible to construct a continuous distribution of temperature and entropy, in space, which would satisfy the fundamental equation (XII-1).

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As Rayleigh has shown [79], this difficulty points to the necessity of taking viscosity into account. However when viscosity is taken into account not only the energy equation but the equation of motion as well [our equation (VIII-2)] is changed. Thus in this case the trajectory of the system in the p, v plane deviates from the line AB. Later these same considerations, with no reference to Rayleigh, were presented by Becker [38] (quoting a private communication of Prandtl; see also [76]).

In the second limiting case, when heat conductivity is absent and only viscosity acts, the change of entropy in the wave takes place only due to the conversion of work against friction into heat [see equation (I-18)].

$$\rho T \frac{dS}{dt} \sim \eta \left(\frac{\partial u}{\partial x} \right)^2. \quad (\text{XII-5})$$

In accordance with the last equation the entropy increases monotonically under the action of viscosity. The change of state in the p, v diagram is represented by a curve which lies between the Poisson curves passing through the initial and final point (broken line in Figure 32). We introduce again the concept of the effective thickness:

$$\frac{\partial u}{\partial x} = \frac{u_B - u_A}{\Delta x}. \quad (\text{XII-6})$$

$$\frac{dS}{dt} = D \frac{\partial S}{\partial x} = D \frac{S_B - S_A}{\Delta x}. \quad (\text{XII-7})$$

Noting that $u_B - u_A = D \cdot \Delta v / v$, we easily find from equation (XII-5) (identifying D and C by their order of magnitude):

$$\Delta x \approx \frac{\eta v^2}{\Delta v C}. \quad (\text{XII-8})$$

The departure of the state from the straight line AB takes place due to the impulse of the viscous forces. The equation for

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one-dimensional steady motion is:

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} - \frac{d}{dx} \left(\frac{2}{3} \eta \frac{du}{dx} \right). \quad (\text{XII-9})$$

Integrating, we find,¹

$$\rho + \rho u^2 + \frac{2}{3} \eta \frac{du}{dx} = p_A + \rho_A u_A^2 = p_B + \rho_B u_B^2, \quad (\text{XII-10})$$

However, from the equation of continuity we have:

$$\rho u = \frac{\rho}{v} = M = \text{const}; \quad \frac{du}{dx} = M \frac{dv}{dx}, \quad (\text{XII-11})$$

$$\rho + Mv + \frac{2}{3} \eta M \frac{dv}{dx} = p_A + Mv_A = p_B + Mv_B = \text{const}. \quad (\text{XII-12})$$

Without the term $\frac{2}{3} \eta M \frac{dv}{dx}$ the equation is that of the straight line AB.

If, according to Figure 32, $\left(\frac{\partial^2}{\partial v^2}\right)_s > 0$, then the broken line between the curves $S = S_A$ and $S = S_B$ lies entirely below the straight line. Therefore, in the wave,

$$\rho + Mv < p_A + Mv_A \quad (\text{XII-13})$$

In this case, from the equation, we find that $\eta M \frac{dv}{dx} < 0$. In the wave v decreases and compression takes place. A rarefaction wave would have required a negative viscosity. Our investigation of the wave front, based on the consideration of viscosity effects, has lead us to the same conclusion as before regarding the relation between the sign of $\left(\frac{\partial^2}{\partial v^2}\right)_s$ and the existence of compression or rarefaction waves.

When heat conductivity is completely absent, the decrease in the coefficient of viscosity will produce only a decrease in the thickness of the front. Therefore the derivative $\frac{du}{dx}$ will increase,

1

Obviously in states A and B, $\frac{du}{dx} = 0$. In the integration we should note that $\rho u = \text{const}$ from the equation of continuity of mass.

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the product $\eta \frac{du}{dx}$ will remain constant and the path in the plane will remain unchanged.

When heat conductivity is present, the decrease in thickness and increase in derivative (with respect to x) will be limited. When η is sufficiently small the entire term $\eta \frac{du}{dx}$ will be small. We shall approach the equation $\rho + Mv = \text{const}$, i.e. the equation of a straight line (see also the remarks made above concerning strong shock waves for which the temperature has a maximum on a segment of line AB. In this case, at some definite part of the wave front, it is precisely the viscosity, no matter how small, that determines the magnitude of the derivatives).

To determine the order of magnitude of the thickness we make use of molecular-kinetic expressions for the coefficients of viscosity and heat conductivity. We can easily find, in both limiting cases:

$$\Delta x \sim l \frac{v}{\Delta v} \sim l \frac{\rho}{\Delta \rho} \sim l \frac{c}{u_1 - u_2}, \quad (\text{XII-14})$$

where l is the mean free path of the molecule in the gas.¹

For air at atmospheric pressure and taking the Prandtl number equal to 1, Taylor [93], [24] gives an expression for the thickness of a shock wave in terms of the coefficient of diffusion B.

$$\Delta x = \frac{4.4 B}{u_1 - u_2}. \quad (\text{XII-15})$$

1

In all of the calculations presented above we have considered an ideal gas for which (at least as regards the order of magnitude) we have the following expression:

$$\left(\frac{\partial \rho}{\partial v} \right)_s \simeq -\frac{\rho}{v}; \quad \frac{\partial^2 \rho}{\partial v^2} \simeq \frac{\rho}{v^2}.$$

In the general case it is easy to establish, that, when all other conditions are equal, the thickness of the front is inversely proportional to the quantity $\left(\frac{\partial^2 \rho}{\partial v^2} \right)_s$, in accordance with the role which this quantity plays in the theory of the shock wave.

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For air at atmospheric pressure $B = 0.18 \text{ cm}^2/\text{sec},$

$$\Delta x = \frac{1}{u_1 - u_2} = 4.10^{-5} \frac{1}{\Delta p} (\Delta x - \text{cm}, u - \text{cm/sec}, \Delta p - \text{atm}). \quad (\text{XII-16})$$

All evaluations show consistently that for shock waves of any strength, in which $\Delta v \approx v$ and $\Delta p \approx p$, the thickness of the front is of the order of the mean free path. Under these conditions a detailed analysis of the structure of a shock wave and the application of the differential equations of hydrodynamics lose their meaning.

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